SHEAR FLOW ZONE IN TORSION
OF REINFORCED CONCRETE

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ABSTRACT: Rausch's classical formula overestimates the torsional strength of reinforced concrete members to an unacceptable degree. The error is traced to the incorrect determination of the centerline of the circulating shear flow. The position of the centerline of shear flow is directly related to the thickness of the shear flow zone \( t_d \). The determination of \( t_d \) in torsion is analogous to the determination of the depth of the compression zone in bending. This paper presents a simple theoretical method to calculate \( t_d \) based on the softened truss model theory. The method utilizes the equilibrium and compatibility conditions, as well as a softened stress-strain relationship for concrete struts. Since \( t_d \) is calculated by a rigorous procedure, an accurate torsional strength can be predicted. The prediction of the torsional strengths of 61 beams found in the literature compares extremely well with the test values. In addition, a very simple formula for \( t_d \) is also proposed for the practical design of members subjected to torsion.

INTRODUCTION

The basic formula for calculating the torsional strength of reinforced concrete members was developed by Rausch (1929) using the space truss concept. Unfortunately, Rausch's equation may be unconservative by more than 30% for under-reinforced beams (Hsu 1968a, 1968b). The error is traced to the incorrect determination of the centerline of the circulating shear flow, resulting in the overestimation of the lever arm area \( A_0 \). The correct determination of the centerline of shear flow depends on a logical way to find the thickness of the shear flow zone, \( t_d \).

Since the late 1960s, the truss model theory for shear and torsion has undergone four major developments. First, the introduction of the variable-angle truss model and the discovery of the bending phenomenon in the diagonal concrete struts were made by Lampert and Thurlimann (1968, 1969). Second, compatibility equation was derived by Collins (1973) to determine the angle of the diagonal concrete struts. Third, the softening phenomenon in the concrete struts was discovered by Robinson and Demorieux (1972), and this behavior was quantified by Vecchio and Collins (1981), using a softening coefficient. Fourth, combining the equilibrium, compatibility and softened stress-strain relationships, a softened truss model theory was developed (Hsu 1988), which was able to analyze the shear and torsional behavior of reinforced concrete members throughout the post-cracking loading history.

Using the softened truss model theory, the thickness of the shear flow zone \( t_d \) can expeditiously be calculated for the torsional strength of reinforced concrete members. This method is presented in this study. In addition, a simple expression for \( t_d \) is proposed for practical design.
ANALOGY BETWEEN TORSION AND BENDING

A prismatic member of arbitrary cross section subjected to torsion is shown in Fig. 1(a). The circulating shear stresses are restricted to an outer ring area with a thickness of $t_d$. Within this thickness a shear flow $q$ acts along a certain centerline, $s$. Taking equilibrium about the center of twist $O$, the external torque $T$ is resisted by the internal moment:

$$ T = q\delta_a ds $$

(1)

where $a$ = the lever arm of shear flow $q$ measured from the center of twist, $O$. The product $\delta_a ds$ is represented graphically by twice the shaded triangular area shown. Therefore, $\delta_a ds$ is twice the area within the centerline of shear flow, and will be denoted as $2A_0$. $A_0$ will be called the lever arm area and is proportional to the square of the level arm $a$. Substituting $2A_0$ into Eq. 1 gives

$$ T = q(2A_0) $$

(2)

Eq. 2 is Bredt’s thin-tube theory (1896), but it should also be applicable to a thick tube, if the position of the centerline of shear flow can be determined.

In a reinforced concrete member, after cracking, as shown in Fig. 2(a), an element isolated from the tube defined by the shear flow zone with a thickness $t_d$, Fig. 2(c), can be represented by a truss model in Fig. 2(d). The element has a vertical length as well as a horizontal length of unity. The diagonal lines representing the cracks are inclined at an angle $\alpha$. Taking equilibrium of forces on the horizontal face and assuming yielding of steel gives:

$$ q = \frac{A_0 f_{sy}}{s} \cot \alpha $$

(3)

![Diagram of Analogy Between Torsion and Bending](image_url)

FIG. 1. Analogy Between Torsion and Bending
Inserting $q$ from Eq. 3 into Eq. 2:

$$T_n = \frac{A_sf_{yy} \cot \alpha}{s} (2A_0) \quad (4)$$

Eq. 4 is the fundamental equation for torsion in the variable-angle truss model. It reduces to the well-known Rausch’s equation (1929) when $\alpha$ is taken as 45°. When equilibrium of forces is taken on the vertical face of the truss model element in Fig. 2(d), $q$ and $T_n$ can be expressed in terms of the longitudinal steel, i.e., $q = (A_sf_{yy}/p_0) \tan \alpha$ and $T_n = [(A_sf_{yy}/p_0) \tan \alpha](2A_0)$.

The analysis of torsion shown is analogous to the analysis of bending in a prismatic members shown in Fig. 1(b). Taking the moment about the centroid of the tension steel, the external moment, $M$, is resisted by the internal moment

$$M = \int (\sigma y) dy = Cjd \quad (5)$$

where $C$ is the resultant of stresses $\sigma$ in the compression zone and $jd = \text{the lever arm of the resultant}$. Comparing Eq. 5 to Eq. 2, the term of twice the lever arm area $2A_0$ in Eq. 2 is equivalent to the resultant lever arm $jd$ in Eq. 5, and the shear flow $q$ is similar to the resultant of compressive stresses $C$.

After cracking of the flexural member, the truss model concept in bending is reduced to the so-called internal couple concept. Assuming the yielding of steel gives

$$C = A_sf_{yy} \quad (6)$$

Substituting $C$ from Eq. 6 into Eq. 5:

$$M_n = A_sf_{yy}(jd) \quad (7)$$

Eq. 7 shows that the bending moment capacity, $M_n$, is equal to the longitudinal steel force, $A_sf_{yy}$, times the resultant lever arm, $jd$. Similarly, in Eq.
the torsional moment capacity, $T_n$, is equal to a certain stirrup force per unit length, $(A_v f_v/s) \cot \alpha$, times twice the lever arm area, $2A_0$.

In bending, an increase of the nominal bending strength $M_n$ due to increasing reinforcement results in an increase of the depth of the compression zone $c$, and a reduction of the resultant lever arm $jd$. The relationships among $M_n$, $c$ and $jd$ can be derived from the stress and strain diagrams. Similarly, in torsion, an increase of the nominal torsional strength $T_n$ due to increasing reinforcement results in an increase of the thickness of shear flow zone $t_d$ and a reduction of the lever arm area $A_0$. The relationships among $T_n$, $t_d$ and $A_0$ can also be derived from the stress and strain diagrams. The understanding of these relationships is a purpose of this study. The crucial problem in torsion of reinforced concrete is to find the thickness of the shear flow zone $t_d$, which is analogous to finding the depth of the compression zone $c$ in bending.

**Various Definitions of Lever Arm Area $A_0$**

When Rausch derived Eq. 4 (with $\alpha = 45^\circ$) in 1929, a reinforced concrete member after cracking was idealized as a space truss. The longitudinal and hoop bars are assumed to take tension and the diagonal concrete struts are in compression. Each diagonal concrete strut is idealized as a straight line lying in the center surface of the hoop bars. Hence, the lever arm area, $A_0$, is defined by the area within the center surface of the hoop bars. This area is commonly denoted as $A_1$. It has been adopted since 1958 by the German Code, and others. Using the bending analogy, this definition is equivalent to assuming that the resultant lever arm, $jd$, is defined as the distance between the centroid of the tension bars and the centerline of the stirrups in the compression zone. In terms of torsional strength this assumption is acceptable near the lower limit of the total steel percentage of about 1%, but becomes increasingly unconservative with an increasing amount of steel (Fig. 3). For a large steel percentage of 2.5–3% near the upper limit of under reinforcement (both the longitudinal steel and stirrups reach yielding), the over prediction of torsional strength by Rausch’s equation using $A_1$ exceeds 30%. This large error is caused by two conditions. First, the thickness of the shear flow zone $t_d$ may be very large, in the order of 1/4 of the outer cross-sectional dimension, due to the softening of concrete (to be explained in the next section). Second, in contrast to the bending strength $M_n$, which is linearly proportional to the resultant lever arm $jd$, the torsional strength $T_n$ is proportional to the lever arm area $A_0$, which, in turn, is proportional to the square of the lever arm, $a$, [Fig. 1(a)].

In order to reduce the unconservatism of using $A_1$ in Rausch’s equation, Lampert and Thurlimann (1969) have proposed that $A_0$ be defined as the area within the polygon connecting the centers of the corner longitudinal bars. This area is commonly denoted as $A_2$ and has been adopted by the CEB-FIP Code (“Model Code” 1978). In terms of the bending analogy, this definition is equivalent to assuming that the resultant lever arm, $jd$, is defined as the distance between the centroid of the tension bars and the centroid of the longitudinal compression bars. The introduction of $A_2$ has reduced the unconservatism of Rausch’s equation for high steel percentages. However, the assumption of a constant lever arm area (not a function of the thickness of shear flow zone) remains unsatisfactory.

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Another way of modifying Rausch’s equation has been suggested by the writer (Hsu 1968a, 1968b) and adopted by the ACI Building Code (“Building Code” 1971).

\[ T_n = T_c + \frac{A_rf_p}{s} (\alpha, A_1) \]  

(8)

where \( \alpha = 0.66 + 0.33 \frac{y_1}{x_1} \leq 1.5; x_1 = \) shorter center-to-center dimension of a rectangular closed stirrup; \( y_1 = \) longer center-to-center dimension of a rectangular closed stirrup; \( T_c = \) nominal torsional strength contributed by concrete \( \approx 0.8x^2y\sqrt{f_c} \) where \( x \) and \( y \) are the shorter and longer sides, respectively, of a rectangular section.

Two modifications of Rausch’s equation are made in Eq. 8 based on tests. First, a smaller lever arm area \( (\alpha/2)A_1 \) is specified, where \( \alpha \) varies from 1 to 1.5. Second, a new term \( T_c \) is added. This term represents the vertical intercept of a straight line in the \( T_n \) versus \( (A_rf_p/s)(A_1) \) diagram (Fig. 3).

Although the addition of \( T_c \) allows the test curve to be closely approximated by a straight line in the under-reinforced region, the complexity that is generated by \( T_c \) is certainly undesirable.

The definitions of the lever arm areas, \( A_1, A_2 \) or \( (\alpha/2)A_1 \), all have a common weakness. They are not related to the thickness of the shear flow zone or the applied torque. A logical way to define \( A_0 \) must start with the determination of the thickness of shear flow zone.

**ANALYSIS OF THICKNESS OF SHEAR FLOW ZONE, \( t_d \)**

The determination of \( t_d \) requires three equations, derived from compatibility, equilibrium, and material law. The derivations are shown.
Compatibility Equations

When a hollow reinforced concrete member is subjected to torsion as shown in Fig. 2, each cross section will rotate, producing an angle of twist, \( \theta \), in the member and a shear strain \( \gamma_h \) in the shear flow tube. According to Bredt’s theory for circulatory torsion (see Chapter 1 in Hsu 1984) \( \theta \) and \( \gamma_h \) are related by the compatibility condition:

\[
\theta = \frac{p_0}{2A_0} \gamma_h
\]

where \( p_0 \) = the perimeter of the centerline of shear flow.

After diagonal cracking and the formation of the truss action, the shear strain \( \gamma_h \) in the shear flow tube will cause tensile strains in the longitudinal and transverse reinforcement \( \epsilon_l \) and \( \epsilon_r \) in the \( l-t \) direction and the principal compressive strains \( \epsilon_d \) and \( \epsilon_r \) in the \( d-r \) direction. The angle between the \( l \) axis and the \( d \) axis is denoted as \( \alpha \), Fig. 2(b). The shear strain \( \gamma_h \) can be expressed in terms of \( \epsilon_d \), \( \epsilon_r \), and \( \alpha \) (Hsu 1988) as follows:

\[
\gamma_h = 2(\epsilon_d - \epsilon_r) \sin \alpha \cos \alpha
\]

In addition to the strain \( \epsilon_d \) in the \( d \)-direction, a diagonal concrete strut will also be subjected to a bending action resulting from the angle of twist \( \theta \), (Fig. 4). The plane OABC lying in the shear flow tube through the centerline of the shear flow is isolated in Fig. 4(b). After twisting, this plane becomes a hyperbolic paraboloid surface OADC. The diagonal line OB, which represents a concrete strut with an angle of inclination \( \alpha \), becomes a curve OD. The curvature of the concrete struts, \( \psi \), can be related by geometry to the
angle of twist, \(\theta\), (Lampert and Thurlimann 1968; Hsu 1984) by:
\[
\psi = \theta \sin 2\alpha 
\]

It should be noted that Eq. 11 is applicable not only to a rectangular section, but also to arbitrary bulky sections. In Fig. 4 a rectangular section is selected to demonstrate the bending curvature in a diagonal concrete strut due to twisting. This is because the imposed curvature is easier visualized in a plane than in a curved surface.

Due to the bending of the diagonal concrete struts the tension area in the inner portion of the cross section will be neglected, Figs. 4 and 5. The compression area will be considered as effective. The depth of the compression zone is denoted \(t_d\), which is identical to the thickness of the shear flow zone. Within the thickness \(t_d\), the strain distribution is assumed to be linear. Therefore, \(t_d\) can be related to the maximum strain at the surface \(\varepsilon_{ds}\) and the curvature \(\psi\) by

\[
t_d = \frac{\varepsilon_{ds}}{\psi} 
\]

Moreover, because of the linear strain distribution, the maximum strain at the surface \(\varepsilon_{ds}\) should be related to the average strain \(\varepsilon_d\) by

\[
\varepsilon_{ds} = 2\varepsilon_d 
\]

The thickness of the shear flow zone \(t_d\) in Eq. 12 can be expressed in terms of \(\varepsilon_d\), \(\varepsilon_r\), and \(\alpha\) by a series of substitution: (1) Substitute \(\gamma \) from Eq. 10 into Eq. 9; (2) substitute \(\theta\) from Eq. 9 into Eq. 11; (3) substitute \(\psi\) from Eq. 11 into Eq. 12; and (4) substitute \(\varepsilon_{ds}\) from Eq. 13 into Eq. 12. The resulting expression is:

\[
t_d = \frac{A_0}{p_0 \sin^2 \alpha \cos^2 \alpha} \left(\frac{\varepsilon_d}{\varepsilon_d - \varepsilon_r}\right) 
\]

Notice that \(A_0\) and \(p_0\) in Eq. 14 are also functions of \(t_d\):

\[
A_0 = A_c - \frac{t_d}{2} p_c + \xi t_d^2
\]

\[
p_0 = p_c - 4\xi t_d
\]
where \( A_c \) = area bounded by the outer perimeter of concrete cross section; 
\( p_c \) = outer perimeter of concrete cross section; \( \xi \) = coefficient equal to 1 for rectangular section and \( \pi/4 \) for circular section. \( \xi \) can be taken as unity for all shapes of cross sections with only negligible loss of accuracy for \( A_0 \) and \( p_0 \).

\( A_0 \) and \( p_0 \) have been expressed in Eqs. 15 and 16 by assuming that the centerline of the shear flow lies midway in the thickness of the shear flow zone \( t_d \). This assumption provides two advantages: First, it simplifies the expressions of \( A_0 \) and \( p_0 \). Second, it is slightly on the conservative side and is desirable to compensate for the slight unconservatism inherent in using the term \( A_s f_y / s \) in Eq. 4 to express the stirrup force per unit length. The effect of the stirrup spacing, \( s \), on the shear strength has been carefully explained in Section 4.4.3.1 of Hsu (1984). It should also be pointed out that \( A_0 \) and \( p_0 \) in Eqs. 15 and 16 are applicable to a thick tube even when \( \xi \) is taken as unity.

Material Law

Being subjected to axial stress and bending, the distribution of the compressive stresses in a diagonal concrete strut within the thickness \( t_d \) is shown by the solid curve in Fig. 5. This stress-strain relationship is based on a softened stress-strain curve, Fig. 6, proposed by Vecchio and Collins (1981). Their concrete test panels were reinforced in both the longitudinal and transverse directions and were subjected to pure shear at the edges. Their tests clearly show that after diagonal cracking the stress and the strain in the concrete struts, \( \sigma_d \) and \( \epsilon_d \), are softened by the tensile strain in the perpendicular direction, \( \epsilon_r \). The softened coefficient \( \zeta \) is a function of the ratio \( \epsilon_d / \epsilon_r \):

\[
\zeta = \sqrt{\frac{\epsilon_d}{\epsilon_d - \epsilon_r}} \tag{17}
\]

The softening coefficient \( \zeta \), which is less than unity, is the reciprocal of the coefficient \( \lambda \) given by Vecchio and Collins (1981). In their paper, \( \epsilon_d \) in the denominator is multiplied by a constant \( (1 - \mu) \), where \( \mu \) is Poisson’s ratio for concrete. The omission of \( \mu \) produces negligible difference (Hsu 1984). Note also that \( \epsilon_d \) is negative and \( \epsilon_r = \epsilon_l - \epsilon_d \) is positive.

Based on the softened stress-strain relationship in Fig. 6, the peak stress is \( \zeta f'_c \) and the average compressive stress, \( \sigma_d \), can be defined as
where \( k_1 \) = the ratio of the average stress to peak stress in the stress block. The \( k_1 \)-ratio can be obtained by integrating the stress-strain curve in Fig. 6 and has been tabulated in Table 7.4 of Hsu (1984) as a function of the maximum strain \( \epsilon_{du} \) and the softening coefficient \( \zeta \). For under-reinforced members, the maximum torque occurs when the maximum concrete strain \( \epsilon_{du} \) varies from 0.0015 to 0.0030, and \( \zeta \) varies from 0.35 to 0.50. Within those ranges, the table shows that \( k_1 \) varies in a narrow range from 0.85 to 0.77. Taking an average value of \( k_1 = 0.80 \), and treating \( f'_c \) as positive, then \( \sigma_d \) becomes

\[
\sigma_d = -0.80 \zeta f'_c \quad \text{(19)}
\]

Substituting the softening coefficient \( \zeta \) from Eq. 17 into Eq. 14, \( t_d \) can be expressed in terms of \( \zeta \) and \( \alpha \):

\[
t_d = \frac{A_\zeta \zeta^2}{p_0 \sin^2 \alpha \cos^2 \alpha} \quad \text{(20)}
\]

It is interesting to note that \( t_d \) in Eq. 20 is no longer a function of the strains \( \epsilon_d \) or \( \epsilon_r \). Physically, this means that \( t_d \) is independent of the loading history.

The substitution of the softening coefficient \( \zeta \) from Eq. 17 into Eq. 14 involves an assumption. Since Eq. 17 is obtained from tests of reinforced concrete panels subjected to pure shear alone, the strains \( \epsilon_d \) and \( \epsilon_r \) in this equation represent the uniform in-plane strains of an element without bending. By contrast, Eq. 14 is derived from an element in a concrete strut subjected to in-plane strains as well as bending, so that \( \epsilon_d \) and \( \epsilon_r \) represent the average strains in the mid depth of the thickness \( t_d \). Therefore, Eq. 20 is obtained by assuming that the softening of a concrete strut subjected to compression and bending is identical to the softening of a concrete strut subjected to the average compression strain without bending. This assumption has yet to be proven by tests, but it should provide a very good approximation.

The thickness of shear flow zone, \( t_d \), can be solved by Eq. 20 in conjunction with two equilibrium equations.

**Equilibrium Equations**

From the truss model of a reinforced concrete element shown in Fig. 2(d) it can be demonstrated that the stresses in the concrete satisfy Mohr’s stress circle (Hsu 1984). Assuming that the steel will yield at failure (for under-reinforced members) and the concrete cannot resist tension in the direction perpendicular to the cracks, i.e., \( \sigma_r = 0 \), then the superposition of the concrete stresses and steel stresses gives the following three equilibrium equations:

\[
\sigma_l = \sigma_d \cos^2 \alpha + \rho_t f_y \quad \text{(21)}
\]

\[
\sigma_t = \sigma_d \sin^2 \alpha + \rho_t f_y \quad \text{(22)}
\]

\[
\tau_{lt} = \sigma_d \sin \alpha \cos \alpha \quad \text{(23)}
\]

where \( \sigma_l, \sigma_t = \text{normal stress in the } l \text{ and } t \text{ directions, respectively (positive for tension); } \tau_{lt} = \text{shear stresses in the } l-t \text{ coordinate (negative, as shown in}
Fig. 2); $p_l, p_t = \text{reinforcement ratio in the } l \text{ and } t \text{ directions, respectively.}$

$p_l = A_l / p_0 t_d$, where $A_l$ is the total area of longitudinal steel in the cross

section; and $p_t = A_t / s t_d$, where $A_t$ is the area of one leg of a hoop bar and $s$ is the spacing of the hoop bars; $f_{l y}, f_{t y} = \text{yield strength of the longitudinal and transverse steel, respectively.}$

For the case of pure torsion, $\sigma_t = \sigma_v = 0$. Adding Eqs. 21 and 22 and inserting $\sigma_d = -0.80 f' c$ result in:

$$\zeta = \frac{A_l f_{l y}}{p_0} + \frac{A_t f_{t y}}{s} \frac{0.80 f' c t_d}{(24)}$$

Substituting $\sigma_d = -0.80 f' c$ from Eq. 19 into Eq. 21 and utilizing Eq.

$$\cos^2 \alpha = \frac{A_l f_{l y}}{p_0} + \frac{A_t f_{t y}}{s} \frac{(25)}$$

Solution Method

The compatibility Eq. 20 and the two equilibrium Eqs. 24 and 25 provide three equations involving three unknown variables, $t_d, \zeta,$ and $\alpha$. The solution of these three simultaneous equations can be obtained by a simple trial-and-error procedure as follows:

1. Assume an initial value of $t_d$ and calculate $A_0$ and $p_0$ by Eqs. 15 and 16.

2. Compute $\zeta$ and $\alpha$ from Eqs. 24 and 25, respectively.

3. Substituting $\zeta$ and $\alpha$ into Eq. 20 gives $t_d$. If the resulting $t_d$ is close enough to the initial value, then a solution with a set of $t_d, \zeta,$ and $\alpha$ values are obtained. If the resulting $t_d$ is not close enough to the initial value, assume another $t_d$, and repeat the cycle.

Once a solution is obtained, the ultimate shear stress $\tau_{ul}$ can be calculated from Eqs. 23 and 19, the torsional strength $T_n$ can be obtained from Eq. 4. An example problem showing the solution procedures is given in Appendix I.

Comparison with Tests

This method of calculating the thickness of shear flow zone and the torsional strength has been applied to analyze the 61 test beams available in literature (McMullen and Warwaruk 1967; Hsu 1968a; Lampert and Thurlimann 1968, 1969; Bradburn 1968; Leonhardt and Schelling 1974; Mitchell and Collins 1974; McMullen and Rangan 1978). The calculated $t_{d,calc}$ and $T_{n,calc}$ are recorded in Table 1. The $T_{n,calc}$ values are also compared to the test values $T_{n,test}$. The average $T_{n,test}/T_{n,calc}$ is 1.010 and the standard deviation is 0.051.

The 61 beams available in literature satisfy the following four criteria (Hsu and Mo 1985a, 1985b): (1) The member should have sufficient reinforce-
<table>
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<th>Specimen</th>
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<th>$T_{n,calc}$ (in.-kip)</th>
<th>$T_{n,brk}$ (in.-kip)</th>
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</tr>
<tr>
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<tr>
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<th>$T_{n,calc}$ (in.-kip)</th>
<th>$T_{n,brk}$ (in.-kip)</th>
<th>$T_{n,brk}/T_{n,calc}$</th>
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<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
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<tbody>
<tr>
<td>(d) Lampert and Thurlimann (1968)</td>
<td></td>
<td></td>
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<td>T0</td>
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<tr>
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<td>1,147.4</td>
<td>1,145.0</td>
<td>0.998</td>
</tr>
</tbody>
</table>

(e) Mitchell and Collins (1974)

| (f) Bradburn (1968) |   |   |   |   |
| PT4 | 2.980 | 589.3 | 620.0 | 1.052 |

(g) McMullen and Warwaruk (1967)

| (g) |   |   |   |   |
| 2-1 | 1.665* | 189.5* | 181.0 | 0.955 |

*Since actual steel stresses in these specimens are slightly less than yield stresses, calculated values are used in computing these values. 1 in. = 25.4 mm; 1 in.-kip = 113 N-m.

The theory presented here has been rigorously derived. The only major inaccuracy introduced is the approximation of \( k_1 = 0.80 \). This approximation should be quite good after the maximum fiber strain \( \varepsilon_{ds} \) reaches well into the descending branch of the softened stress-strain curve, i.e. \( \varepsilon_{ds} > 1.5\varepsilon_0 \) where \( \varepsilon_0 \) is taken as 0.002. Therefore, the theory is very suitable for finding the torsional strength. At the low load stages when \( \varepsilon_{ds} < \xi_0 \), however, \( k_1 = 0.80 \) would not be sufficiently accurate, and a more general method of solution (Hsu 1988) should be used.

**THICKNESS OF SHEAR FLOW ZONE FOR DESIGN**

The thickness of the shear flow zone given in Eq. 20 is suitable for the analysis of torsional strength. It is, however, not convenient for the design of torsional members. In design, the thickness of shear flow zone \( t_d \) should be expressed in terms of the torsional strength, \( T_n \). This approach will now be introduced.

The stress in the diagonal concrete struts, \( \sigma_d \), can be related to the thickness \( t_d \) and the shear flow \( q \) using the equilibrium Eq. 23:

\[
\sigma_d = \frac{q}{t_d \sin \alpha \cos \alpha} \tag{26}
\]
At failure, $\sigma_d$ in Eq. 26 reaches the maximum $\sigma_{d,\text{max}}$, while the torsional moment reaches the nominal capacity $T_n$. Substituting $q = T_n/2A_0$ at failure into Eq. 26 gives:

$$t_d = \frac{T_n}{2A_0 \sigma_{d,\text{max}} \sin \alpha \cos \alpha} \quad (27)$$

Assuming $t_d$ to be thin, the last term $\xi t_d^2$ in Eq. 15 is neglected, and $A_0$ can be expressed by the thin-tube approximation:

$$A_0 = A_c - \frac{t_d}{2} p_c \quad (28)$$

Substituting $A_0$ from Eq. 28 into Eq. 27 and multiplying all the terms by $2p_c/A_c^2$ results in:

$$\left( \frac{p_c}{A_c} \right)^2 - 2 \left( \frac{p_c}{A_c} t_d \right) + \frac{T_n p_c}{A_c^2} \frac{1}{\sigma_{d,\text{max}} \sin \alpha \cos \alpha} = 0 \quad (29)$$

Define:

- $t_{d0} = A_c/p_c$
- $\tau_n = T_n p_c / A_c^2$
- $\tau_{n,\text{max}} = \sigma_{d,\text{max}} \sin \alpha \cos \alpha$

Eq. 29 becomes

$$\left( \frac{t_d}{t_{d0}} \right)^2 - 2 \left( \frac{t_d}{t_{d0}} \right) + \frac{\tau_n}{\tau_{n,\text{max}}} = 0 \quad (30)$$

When $t_d/t_{d0}$ is plotted against $\tau_n/\tau_{n,\text{max}}$ in Fig. 7, Eq. 30 represents a parabolic curve. Solving $t_d$ from Eq. 30 gives:

$$t_d = t_{d0} \left[ 1 - \sqrt{1 - \frac{\tau_n}{\tau_{n,\text{max}}}} \right] \quad (31)$$

This approach of determining the thickness of the shear flow zone, first proposed by Collins and Mitchell (1980) and later adopted by the Canadian Code (“Design” 1984), gives:

$$t_d = \frac{A_1}{p_1} \left[ 1 - \sqrt{1 - \frac{T_n p_1}{0.7 \phi_c f' c A_1^2} \left( \tan \alpha + \frac{1}{\tan \alpha} \right)} \right] \quad (32)$$

In Eq. 32 $A_c$ and $p_c$ are replaced by $A_1$ and $p_1$, respectively, since the concrete cover is considered ineffective. $\sigma_{d,\text{max}}$ is assumed to be $0.7 \phi_c f' c$, in which the material reduction factor $\phi_c$ can be taken as 0.6.

Eqs. 32 and 31 clearly show that the thickness ratio, $t_d/t_{d0}$, is primarily a function of the shear stress ratio, $\tau_n/f' c$. The thickness ratio $t_d/t_{d0}$ is also a function of the crack angle $\alpha$, but is not sensitive when $\alpha$ varies in the vicinity of 45°.

Eq. 31, $\tau_n < \tau_{n,\text{max}}$, represents the case of under-reinforcement, while $\tau_n > \tau_{n,\text{max}}$ means over-reinforcement. The case of over-reinforcement cannot be expressed by Eq. 31, because it gives a complex number ($\sqrt{-1}$). Fig.
FIG. 7. Graphical Presentation of Eqs. 30 or 31

7 shows that Eq. 31 is applicable when \( \tau_n \) is less than about 0.9 \( \tau_{n,\text{max}} \). However, when \( \tau_n \) exceeds 0.9 \( \tau_{n,\text{max}} \), \( t_d \) is increasing unreasonably fast. This problem reflects the difficulty in using the thin-tube approximation for \( A_0 \) (Eq. 28) to find \( t_d \). When \( t_d \) exceeds about 0.7 \( t_{d,0} \), the tube becomes so thick that the term \( \xi \) cannot be neglected.

To avoid this weakness, the writer has adopted a different approach. Using the softened truss model theory, a computer program was written to analyze the torsional behavior of reinforced concrete members throughout the loading history (Hsu and Mo 1985a). This computer program was used to analyze the 61 eligible torsional members (satisfying the four criteria previously cited) available in literature. The thicknesses of the shear flow zones in the test beams are calculated from the computer program and a linear regression analysis of the thickness ratios \( t_d/t_{d,0} \) is made as a function of \( \tau_n/f' c \). This analysis provides the following expression (Hsu and Mo 1985b):

\[
\frac{t_d}{t_{d,0}} = \frac{A_c}{p_c} \left( 0.082 + 3.405 \frac{\tau_n}{f' c} \right) \left( \frac{1}{\sin 2\alpha} \right) \tag{33}
\]

Eq. 33 is plotted in Fig. 8 for the cases of \( \alpha = 45^\circ \) and \( \alpha = \tan^{-1}(5/3) \) or \( \tan^{-1}(3/5) \), which are the limits adopted by CEB-FIP Code (“Model Code” 1978). The 61 test points are also included and the correlation is shown to be excellent. The \( t_d/t_{d,0} \) values calculated from Eq. 33 for the writer series B are recorded in Table 2. When compared to the \( t_d/t_{d,0} \) values obtained from the computer program, the correlation is again excellent. The 10 beams in Series B were chosen because they have total reinforcement ratios varying from a low of 1.07% to a high of 5.28%, and a volume ratio of longitudinal steel to stirrups varying from 0.205 to 4.97. The wide range of application of Eq. 33 is evident. It is not only applicable to under-reinforced members, but also to over-reinforced members.

Although Eq. 33 is found to be excellent, it is considered somewhat un-
TABLE 2. Comparison of Thickness Ratios $t_d/t_{d0}$

<table>
<thead>
<tr>
<th>Beam (1)</th>
<th>$\rho_l$ (%) (2)</th>
<th>$\rho_t$ (%) (3)</th>
<th>$T_n$ (in.-kip) (4)</th>
<th>$\alpha$ (°) (5)</th>
<th>$\zeta$ (6)</th>
<th>$t_d$ (in.) (7)</th>
<th>$t_d/t_{d0}$ (8)</th>
<th>$t_d/t_0$ (9)</th>
<th>$t_d$ (10)</th>
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<tbody>
<tr>
<td>B1</td>
<td>0.534</td>
<td>0.537</td>
<td>202</td>
<td>46.5</td>
<td>0.372</td>
<td>1.41</td>
<td>0.470</td>
<td>0.464</td>
<td>0.449</td>
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<tr>
<td>B2</td>
<td>0.827</td>
<td>0.823</td>
<td>287</td>
<td>44.8</td>
<td>0.340</td>
<td>1.78</td>
<td>0.593</td>
<td>0.606</td>
<td>0.615</td>
</tr>
<tr>
<td>B3</td>
<td>1.77</td>
<td>1.17</td>
<td>370</td>
<td>44.6</td>
<td>0.510</td>
<td>2.32</td>
<td>0.770</td>
<td>0.770</td>
<td>0.808</td>
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<tr>
<td>B4</td>
<td>1.60</td>
<td>1.61</td>
<td>431</td>
<td>44.6</td>
<td>0.531</td>
<td>2.46</td>
<td>0.820</td>
<td>0.818</td>
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<td>2.16</td>
<td>0.720</td>
<td>0.691</td>
<td>0.648</td>
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</table>

Note: $\rho_l$ and $\rho_t$ are the reinforced ratios of longitudinal steel and transverse hoop steel, respectively, based on total cross-sectional area $A_c$. Cross section 10 in. x 15 in., $f_y = 47,000$ psi; $f'_c = 4,000$ psi; 1 in. = 25.4 mm; 1 psi = 6.89 kPa.

FIG. 8. Thickness Ratio $t_d/t_{d0}$ as Straight Line Functions of Shear Stress Ratio, $\tau_n/f'_c$.

Wieldy for practical design. In the next section a simplified expression for $t_d$ is proposed. The simplicity is obtained with a small sacrifice in accuracy.

PROPOSED $t_d$ FOR DESIGN

A simple expression for the thickness of shear flow zone, $t_d$, can be obtained directly from Eq. 2, noting that $q = \tau_n t_d$ and $T = T_n$:

$$ t_d = \frac{T_n}{2A_0\tau_0} \tag{34} $$

Assuming that $A_0 = m_1A_c$ and $\tau_0 = m_2f'_c$ where $m_1$ and $m_2$ are non-dimensional coefficients, substituting them into Eq. 34 gives

$$ t_d = \frac{C}{A_c f'_c} \frac{T_n}{T_0} \tag{35} $$

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where \( C = \tfrac{1}{2} m_1 m_2 \). For under-reinforced members, \( m_1 \) varies from 0.55 to 0.85, while \( m_2 \) varies from 0.13 to 0.22. These values are obtained from the Appendix of Hsu and Mo’s report (1983). The low values of \( m_2 \) are due to the softening of concrete. For an increasing amount of reinforcement, \( m_2 \) increases while \( m_1 \) decreases. Therefore, the product \( m_1 m_2 \) can be taken approximately as a constant, 0.125, making \( C \) a constant of 4. Then

\[ \text{FIG. 9. Comparison of Test Strengths with Calculated Strengths Using Proposed } \tau_d \text{ (Eq. 36)} \]

\[ \text{FIG. 10. Comparison of Test Strengths with Calculated Strengths Using Proposed } \tau_d \text{ (Eq. 36)—Expanded Scale for Lower Portion of Fig. 9} \]
Eq. 36 is plotted in Fig. 8. Comparison with Eq. 33 shows the difference to be small. Actually, Eq. 36 can be considered a simplification of Eq. 33 by neglecting the small first term with constant 0.082 and increasing the constant in the second term from 3.405 to 4. The small effect of $\alpha$ is also neglected by taking $\sin 2\alpha = 1$, which is the exact value when $\alpha = 45^\circ$. The $t_d/t_{d0}$ ratios calculated from Eq. 36 for the writer's B series are also recorded in Table 2. A comparison with the computer values also shows the correlation to be reasonable.

Inserting Eq. 36 into the thin-tube expression of $A_0$ in Eq. 28 gives:

$$A_0 = A_c - \frac{2T_0p_c}{A_c f_c'} \quad \text{........................................ (37)}$$

$A_0$ in Eq. 37 is used in conjunction with Eq. 4 to calculate the torsional strength, $T_n$, for the 61 beams available in literature. The calculated values are compared to the test values in Figs. 9 and 10. The average $T_{n,\text{test}}/T_{n,\text{calc}}$ value is 1.013 and the standard deviation is 0.055.

An example problem is given in Appendix II showing the procedures for designing reinforced concrete members subjected to torsion.

**ACKNOWLEDGMENT**

The writer would like to express his deep appreciation to National Science Foundation for supporting this study through Grant No. ECE-8511876. He also wishes to thank X. B. Pang, graduate student in the civil engineering department, University of Houston, for the computation of Table 1, and the plotting of Figs. 9 and 10.

**APPENDIX I. ANALYSIS EXAMPLE**

Beam G7 (Hsu 1968a) has a rectangular cross section of 10 in. by 20 in. (25 cm by 50 cm). It is reinforced with 6 No. 5 longitudinal bars (4 at corners and 2 at midheight of longer side) and No. 4 closed stirrups with uniform spacing of 5.75 in. (14.6 cm). The material properties are: $f_{ly} = 46.3$ ksi (319 MPa), $f_{y} = 46.8$ ksi (322 MPa), and $f'_{c} = 4.49$ ksi (30.9 MPa).

Solution:

$$A_c = 10(20) = 200 \text{ sq in. (1,290 cm}^2)$$
$$p_c = 2(10 + 20) = 60 \text{ in. (152.4 cm)}$$
$$A_t = 6(0.31) = 1.86 \text{ sq in. (12.0 cm}^2)$$
$$A_r = 0.20 \text{ sq in. (1.29 cm}^2)$$

Assume $t_d = 2$ in. (5.08 cm)

Substitute into Eq. 15.

$$A_0 = A_c - p_c \left(\frac{t_d}{2}\right) + t_d^2 = 200 - 60 \left(\frac{2}{2}\right) + (2)^2 = 144 \text{ sq in. (929 cm}^2)$$

and into Eq. 16.
\[ p_0 = p_c - 4t_d = 60 - 4(2) = 52 \text{ in. (132 cm)} \]

\[ \frac{A_t f_y}{p_0} = \frac{1.86(46.3)}{52} = 1.656 \text{ kip/in. (290 kN/m)} \]

\[ \frac{A_t f_y}{s} = \frac{0.20(46.8)}{5.75} = 1.628 \text{ kip/in. (285 kN/m)} \]

then Eq. 24.

\[ \zeta = \frac{\left( \frac{A_t f_y}{p_0} \right) + \left( \frac{A_t f_y}{s} \right)}{0.80 f'_c t_d} = \frac{1.656 + 1.628}{0.80(4.49)(2)} = 0.4571 \]

and Eq. 25.

\[ \cos^2 \alpha = \frac{\frac{A_t f_y}{p_0}}{\left( \frac{A_t f_y}{p_0} \right) + \left( \frac{A_t f_y}{s} \right)} = \frac{1.656}{1.656 + 1.628} = 0.5042 \]

\[ \sin^2 \alpha = 0.4958 \]

and finally into Eq. 20.

\[ t_d = \frac{A_0 \xi^2}{p_0 \sin^2 \alpha \cos^2 \alpha} = \frac{144(0.4571)^2}{52(0.5042)(0.4958)} = 2.314 \text{ in. (5.88 cm)} > 2 \text{ in. (5.08 cm)} \]

N.G. repeat cycle assuming \( t_d = 2.10 \text{ in. (5.22 cm)} \)

Now

\[ t_d = 2.094 \text{ in. (5.32 cm)} = 2.10 \text{ in. (5.33 cm)} \text{ o.k.} \]

and finally Eq. 4.

\[ T_{n, \text{calc}} = \frac{A_t f_y \cot \alpha}{s} (2A_0) = \frac{0.20(46.8)(1.013)}{5.75} (2 \times 141.4) = 466.3 \text{ in.-kip (52.69 kN-m)} \]

The experimental torsional strength (\( T_{n, \text{test}} \)) of Beam G7 is found to be 466 in.-kip (52.66 kN-m), which is very close to the calculated value of 466.3 in.-kip (52.69 kN-m). It should be noted that the thickness of the shear flow zone \( t_d \) of 2.10 in. (5.33 cm) is more than 1/5 of the beam width (10 in. or 25.4 cm) for a beam with moderate total reinforcement ratio of 1.87%. It is obvious that \( t_d \) of a reinforced concrete beam could become very big when the reinforcement ratio is large. The definition of the lever arm area \( A_0 \) by a constant \( A_t \) in Rausch's Eq. 4 could therefore produce a large error of the torsional strength on the unconservative side.
**APPENDIX II. DESIGN EXAMPLE**

Design the reinforcement for the hollow box beam with the trapezoidal cross section as shown in Fig. 11. The beam should be able to resist a torsional moment of 7,400 in.-kip (836 kN-m). The net concrete cover is 1.5 in. (3.81 cm) and the material strengths are $f'_c = 4,000$ psi (27.6 MPa) and $f_y = 60,000$ psi (413 MPa).

**Solution**

For the given outer cross-sectional dimensions shown in Fig. 11

\[ A_c = \frac{(3 + 4)(3)(12)^2}{2} = 1,512 \text{ sq in. (9,755 cm}^2)\].

\[ p_c = (3 + 4 + 2\sqrt{3^2 + 0.5^2})(12) = 157 \text{ in. (399 cm)}\].

**Check Cracking Torque**

\[ T_c = A_c t (5\sqrt{f'_c}) = 1,512(5)\sqrt{4,000} = 2,391 \text{ in.-kip (270 kN-m)} \]

\[ T_n = 7,400 \text{ in.-kip (836 kN-m)} > 2,391 \text{ in.-kip (270 kN-m)} \]

Reinforcement required.

**Calculate** $t_d$, $A_0$ and $p_0$

Eq. 36.

\[ t_d = \frac{A_c}{A_c f'_c} \frac{4(7,400)}{1,512} = 4.89 \text{ in. (12.4 cm) < 5 in. (12.7 cm)} \]

wall thickness o.k.

\[ A_0 = A_c - \frac{p_c t_d}{2} = 1,512 - \frac{157(4.89)}{2} = 1,128 \text{ sq in. (7,277 cm}^2) \]

\[ p_0 = p_c - 4t_d = 157 - 4(4.89) = 137.4 \text{ in. (347 cm)} \]

*FIG. 11. Design Example*
Design of Stirrups

Eq. 4.

\[
A_i = \frac{T_n \tan \alpha}{s} = \frac{7,400 \tan \alpha}{2A_0 f_y} = \frac{2(1,128)(60)}{2 \tan \alpha} = 0.0547 \text{ (tan } \alpha \text{) sq in./in. (0.139 (tan } \alpha \text{) cm}^2/\text{cm)}
\]

The minimum \( \alpha \) is

\[
\alpha = 12^\circ + 33^\circ \left[ \frac{T_n P_c}{A_0^2 f'_c (0.27 - 45 \varepsilon)} \right]
\]

\[
= 12^\circ + 33^\circ \left[ \frac{7,400(157)}{(1,512)^2(4)(0.27 - 45 \cdot 0.00207)} \right]
\]

\[
= 35.7^\circ < 45^\circ. \text{ Under-reinforced.}
\]

Select \( \alpha = 45^\circ \) for best crack control.

\[
A_i = \frac{0.0547(1)}{0.0547} = 0.0547 \text{ sq in./in. (0.139 cm}^2/\text{cm)}
\]

Select No. 6 bars \( s = \frac{0.44}{0.0547} = 8.04 \text{ in. (20.4 cm)} \)

Check stirrup spacing

\[
p_1 = \frac{157 - 4(2)(1.5 + \frac{0.75}{2})}{8} = 17.75 \text{ in. (45.1 cm)}
\]

\( s = 8.04 < 12 \text{ in. (30 cm)} < 17.75 \text{ in. (45.1 cm)} \) o.k.

Use No. 6 transverse hoop bars at 8 in. (20.3 cm) spacing.

Design of Longitudinal Steel

\[
A_i = \frac{T_n P_0}{2A_0 f_y \tan \alpha} = \frac{7,400(137.4)}{2(1,128)(60)(1)} = 7.51 \text{ sq in. (48.4 cm}^2) \)

Select 13 No. 7 longitudinal bars so that spacing will be less than 12 in. (30 cm).

Actual \( A_i = 13(0.60) = 7.80 \text{ sq in. (50.3 cm}^2) > 7.51 \text{ sq in. (48.4 cm}^2) \) o.k.

APPENDIX III. REFERENCES

"Bemessung im Stahlbetonbau" (Design of Reinforced Concrete), (1958). German Standard DIN 4334. Wilhelm Ernst and Sohn, Berlin, West Germany (in German).


"Building code requirements for reinforced concrete (ACI 318-83)." (1971). American Concrete Institute, ACI Committee 318, Detroit, Mich.


